Semi-Classical and Functional Methods: Lecture I notes

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In this lecture, we derive the (1PI) effective action from the path integral by way of a suitably defined generating functional. We show how one can account for quantum corrections as a systematic expansion in powers of $\hbar$. We then show how one would formally calculate all contributions (to one loop) in a diagrammatic expansion, moving on to show how one can explicitly evaluate the effective action using the heat kernel formalism. We conclude with a simple application– deriving the Coleman-Weinberg effective potential in flat space.

I. THE EFFECTIVE ACTION FROM THE PATH INTEGRAL.

We begin with the generating functional $Z[j]$ for a scalar field ϕ with the action $S[\phi]$ coupled to an external source j:

$$Z[j] = \int \mathcal{D}\phi \ \mathrm{e}^{\frac{i}{\hbar}[S[\phi]] + \int \mathrm{d}^4x \, j\phi]}, \tag{1}$$

with which we can construct all n-point correlation functions as

$$\langle \phi_1 ... \phi_n \rangle = \frac{1}{Z} \int \mathcal{D}\phi \ \phi_1 ... \phi_n \ \mathrm{e}^{\frac{i}{\hbar} S[\phi]}. \tag{2}$$

Defining W[j] as

$$\mathrm{e}^{\frac{i}{\hbar} W[j]} := Z[j], \tag{3}$$

we assert that $W[j]$ is the generating functional of all connected Green's functions. Although a formal proof of this statement is available in many standard QFT textbooks (e.g. [1]), we content ourselves here with an explicit demonstration up to the three point correlator:

$$\begin{aligned} W[j] &= -i\hbar \, lnZ[j] \\ \frac{\delta W}{\delta j(x_1)} &= -i\hbar \frac{1}{Z} \frac{\delta Z[j]}{\delta j(x_1)} \\ &= \frac{-i\hbar}{Z} \int \mathcal{D}\phi \ \frac{i\phi(x_1)}{\hbar} \mathrm{e}^{\frac{i}{\hbar}[S[\phi]] + \int \mathrm{d}^4x \, j\phi]} \\ &= \frac{1}{Z} \int \mathcal{D}\phi \ \phi(x_1) \mathrm{e}^{\frac{i}{\hbar}[S[\phi]] + \int \mathrm{d}^4x \, j\phi]} \end{aligned} \tag{4}$$

Hence

$$\left. \frac{\delta W}{j(x_1)} \right|_{j\equiv 0} = \langle \phi(x_1) \rangle$$

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The above quantity– the so called quantum averaged field– will play a very important role in what follows. Taking another functional derivative with respect to to the external source, we find:

$$\frac{\delta^2 W}{\delta j(x_1)\delta j(x_2)}\Big|_{j\equiv 0} = \frac{i}{\hbar}[\langle\phi(x_1)\phi(x_2)\rangle - \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle], \tag{5}$$

and similarly again

$$\begin{aligned}\frac{\delta^3 W}{\delta j(x_1)\delta j(x_2)\delta j(x_3)}\Big|_{j\equiv 0} = \left(\frac{i}{\hbar}\right)^2 &[\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle - \langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\rangle \\ &+ perm + 2\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle\langle\phi(x_3)\rangle].\end{aligned} \tag{6}$$

Therefore anecdotally justifying $W[j]$ as the generator of connected correlation functions.

Exercise: Take one more functional derivative to obtain the connected four point correlation function, assuming now that $\langle\phi(x)\rangle \equiv 0$. What would the result evaluate to for a purely Gaussian field?

Specifically defining the C-number field $\Phi(x)$

$$\begin{aligned}\Phi(x) &= \frac{\delta W}{\delta j(x)} \\ &\equiv \langle\phi(x)\rangle,\end{aligned} \tag{7}$$

we now invert this to yield j(x) as x-dependent functional of $\Phi(x)$

$$j(x) = j[\Phi](x). \tag{8}$$

Define now the so-called *effective action* as[1]

$$\Gamma[\Phi] \equiv W[j] - \int d^4y j(y)\Phi(y) \tag{9}$$

where the above consists merely in having made a Legendre transformation to render $\Phi(x)$ the independent (functional) variable. Taking the functional derivative of $\Gamma[\Phi]$ w.r.t. Φ, we see that

$$\begin{aligned}\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} &= \int d^4y \frac{\delta W}{\delta j(y)}\frac{\delta j(y)}{\delta\Phi(x)} - \int d^4y \frac{\delta j(y)}{\delta\Phi(x)}\Phi(y) - j(x) \\ &= -j(x)\end{aligned} \tag{10}$$

where we have used (7). *Therefore the effective action $\Gamma[\Phi]$ is the quantity that generates the quantum corrected equations of motion by being extremized with respect to variations of the (vacuum) expectation values of the fields.* W[j] can be formally recovered from

$$W[j] = \Gamma[\Phi] + \int d^4x j(x)\Phi(x) \tag{11}$$

[1] We justify this definition simply by demonstrating its salient properties in what follows. The treatment in this section closely follows that of [2].

We note that in general, the generating functional is only defined up to an arbitrary normalization factor (that drops out of all observable quantities, i.e. transition amplitudes)

$$Z[j] = \mathcal{N} \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}[S[\Phi] + \int d^4x j \phi]} \tag{12}$$

It is convenient to define the normalization as

$$\mathcal{N} = [\int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S_0[\phi]}]^{-1} \tag{13}$$

where $S_0[\phi]$ is the free (or more generally, solvable) part of the interaction. Therefore

$$e^{\frac{i}{\hbar} W[j]} = \mathcal{N} \; Z[j] \tag{14}$$

can be re-expressed through (11) as

$$e^{\frac{i}{\hbar}\left[\Gamma[\Phi] + \int d^4x \, j(x)\Phi(x)\right]} = \mathcal{N} \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}[S[\Phi] + \int d^4x j \phi]} \tag{15}$$

From the right hand side of the above, we clearly see that $\hbar$ is a measure of the size of quantum fluctuations. From the argument of the exponential, we see that as one integrates over the functional measure, all fluctuations except those for which the stationary approximation is valid are suppressed. In most physical systems, this occurs only in the neighbourhood of a few critical points, whose size is set by $\hbar$. Therefore expanding in powers of $\hbar$ is equivalent to accounting for successively larger quantum fluctuations. Beginning with the zeroth order approximation, at $\lim_{\hbar \to 0}$ path integral is dominated by the classical path

$$\frac{\delta S}{\delta \phi}\Big|_{\phi = \phi_{cl}} = -j(x) \tag{16}$$

Therefore one can equate exponents in (15) (and dropping a trivial constant term that results from making the saddle point approximation)

$$W[j] = \Gamma[\Phi] + \int d^4x j(x)\Phi(x) = S[\phi_{cl}] + \int d^4x \, j(x)\phi_{cl} \tag{17}$$

Taking the functional derivative of the above with respect to $j(x)$

$$\frac{\delta W[j]}{\delta j(x)} = \Phi(x) = \int d^4y \frac{\delta S}{\delta \phi_{cl}} \frac{\delta \phi_{cl}}{\delta j(x)} + \phi_{cl} + \int d^4y \; j(y) \frac{\delta \phi_{cl}}{\delta j(x)} \tag{18}$$

and using (16), we find the zeroth order solution,

$$\Phi(x) = \phi_{cl} \tag{19}$$

so that to zeroth order in $\hbar$, the effective action is the same as the classical action but now evaluated on the quantum averaged field Φ

$$\Gamma[\Phi] = S[\Phi]. \tag{20}$$

Without loss of generality, we consider the functional generalization of the Taylor expansion for the effective action, writing

$$\Gamma[\Phi] = \int ... \int \sum_{n=0}^{\infty} \frac{\Gamma_n}{n!}(x_1, ..., x_n)\Phi(x_1)...\Phi(x_n) \tag{21}$$

where

$$\Gamma_n(x_1, ..., x_n) := \frac{\delta^n \Gamma}{\delta\Phi(x_1)...\,\delta\Phi(x_n)}\bigg|_{\Phi\equiv\phi_{cl}} \tag{22}$$

so that we expand the effective action around the correct vacuum. For example, consider a O(N) ϕ^4 theory, with component fields ϕ^a:

$$S[\phi_{cl}] = \Gamma[\Phi] = \int d^4x\Big[-\frac{1}{2}\partial_\mu\Phi_a\partial^\mu\Phi^a - \frac{m^2}{2}\Phi_a\Phi^a - \frac{g}{4!}(\Phi_a\Phi^a)^2\Big] \tag{23}$$

where we have assumed a flat target space metric, so there is no particular significance to whether the target space index a is raised or lowered. We note that if $m^2 > 0$, there exists an extremum at $\Phi \equiv 0$. Therefore:

$$\begin{aligned}\Gamma^{(2)}(x_1, x_2)_{ab} &= \frac{\delta^2\Gamma}{\delta\Phi_a(x_1)\delta\Phi_b(x_2)} \\ &= (\Box - m^2)\delta_{ab}\delta^4(x_1, x_2),\end{aligned} \tag{24}$$

which determines the inverse propagator

$$\Gamma^{(2)}(x_1, x_2)_{ab} \to \left[i\hbar G^{-1}\right]_{ab}(x_1, x_2). \tag{25}$$

Furthermore

$$\begin{aligned}\Gamma^4_{abcd}(x_1, x_2, x_3, x_4) &= \frac{\delta^4\Gamma}{\delta\Phi_a(x)...\delta\Phi_d(x_4)} \\ &= \frac{g}{3}[\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}].\end{aligned} \tag{26}$$

All other vertex functions vanish. *Using the classical action with quantum average Φ replacing ϕ_{cl} is known as the mean field approximation.* We now consider $\hbar$ corrections. We first parametrize fluctuations around the extremum solution in function space as

$$\phi(x) = \phi_{cl}(x) + \delta\phi(x) \tag{27}$$

so that expanding the argument of the generating functional (15) up to quadratic order in $\delta\phi$ as

$$\begin{aligned}S[\phi] + \int d^4x\, j(x)\phi(x) = S[\phi_{cl}] &+ \int d^4x\, \cancel{\frac{\delta S}{\delta\phi_{cl}(x)}\delta\phi(x)} \\ &+ \int d^4x d^4y\, \frac{1}{2}\frac{\delta^2 S}{\delta\phi_{cl}(x)\phi_{cl}(y)}\delta\phi(x)\delta\phi(y) + ... \\ &+ \int d^4x\, j(x)\phi_{cl}(x) + \int d^4x\, \cancel{j(x)\delta\phi(x)}\end{aligned} \tag{28}$$

Where the cancellation follows from (16). Therefore, the generating functional becomes

$$Z[j] \approx \mathcal{N} \mathrm{e}^{\frac{i}{\hbar}[S[\phi_{cl}]+\int d^4x\, j(x)\phi_{cl}(x)]} \times \int \mathcal{D}\delta\phi \exp\left[\frac{i}{2\hbar}\int d^4x d^4y\, \delta\phi(x)\frac{\delta^2 S}{\delta\phi_{cl}(x)\delta\phi_{cl}(y)}\delta\phi(y)\right] \tag{29}$$

We see that because of the denominator in the Fresnel integral above, the paths that dominate the functional integration will have characteristic fluctuations of the order $\delta\phi \approx O(h^{1/2})$. Therefore in stopping at quadratic order in $\delta\phi$, we are neglecting terms of $O(h^{3/2})$. In this approximation, we have

$$Z[j] \approx \mathcal{N} \mathrm{e}^{\frac{i}{\hbar}[S[\phi_{cl}]+\int d^4x\, j(x)\phi_{cl}(x)]} \det\left[\frac{\delta^2 S}{\delta\phi_{cl}(x)\delta\phi_{cl}(y)}\right]^{-1/2}. \tag{30}$$

A few explanatory words are in order here– the evaluation of the (Gaussian) path integral into a functional determinant can be thought of as the infinite dimensional generalization of the following finite dimensional integral of a quadratic form (x, Ax), with $(\ ,\ )$ an N dimensional scalar product and A some symmetric (more generally, self-adjoint) operator with eigenvalues $\{a_i\}_{i=1}^N$:

$$\int d^N x\, \mathrm{e}^{-\lambda(x,Ax)} = \prod_{i=1}^{N}\sqrt{\frac{\pi}{\lambda a_i}} \equiv \widetilde{\mathcal{N}} \det A^{-1/2}, \tag{31}$$

where $\widetilde{N}$ is another normalization that will not matter in the end. We furthermore note the identity

$$(\mathrm{detA})^{-1/2} = \prod_{i=1}^{N} a_i^{-1/2} = \mathrm{e}^{\log\prod_{i=1}^N a_i^{-1/2}} = \mathrm{e}^{-\frac{1}{2}\sum_i \log a_i} = \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\log A} \tag{32}$$

Therefore

$$\mathcal{N} = \left[\mathcal{D}\phi\ \mathrm{e}^{\frac{i}{\hbar}S_0[\phi]}\right]^{-1} := \left(\det\Omega_0\right)^{1/2}; \quad \text{e.g.}\ \Omega_0 = (\Box - m^2) \tag{33}$$

where more generally Ω_0 is the kernel of the free part of the action.

$$Z[j] = (\det\Omega_0)^{1/2} \exp\left[\frac{i}{\hbar}\left\{S[\phi_{cl}] + \int d^4x j(x)\phi_{cl} + \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}} + ...\right\}\right] \tag{34}$$

Recalling that

$$W[j] = \Gamma[\Phi] + \int d^4x\, j(x)\Phi(x) \tag{35}$$

And making the expansions

$$\begin{aligned} W[j] &= W_0[j] + \hbar W_1[j] + ... \\ \Phi &= \phi_{cl} + \hbar\phi_1 + ... \end{aligned} \tag{36}$$

we expand (15) using (34) and (35) (and ignoring constant normalizations) to find:

$$\begin{aligned} \Gamma[\Phi] + \int d^4x\, \cancel{j(x)\Phi(x)} &= S[\phi_{cl}] + \int d^4x\, j(x)\phi_{cl} + \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}} \\ &= S[\Phi - \hbar\phi_1] + \int d^4x\, j(x)[\cancel{\Phi} - \hbar\phi_1] \\ &\quad + \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\Phi-\hbar\phi_1} \end{aligned} \tag{37}$$

Further expanding $S[\Phi - \hbar\phi_1]$ in the above, we find

$$\begin{aligned}\Gamma[\Phi] &= S[\phi_{cl}] - \hbar[\cancel{\frac{\delta S_{cl}}{\delta\Phi}} + j]\phi_1 + \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\Phi-\hbar\phi} \\ &= S[\phi_{cl}] + \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\Phi=\phi_{cl}} + \mathcal{O}(\hbar^{3/2}),\end{aligned} \tag{38}$$

Where the cancellation above follows from (16) and the fact we only work to $\mathcal{O}(\hbar)$ so that we can take $\Phi = \phi_{cl}$ within the term in the square brackets. Therefore, equating powers of $\hbar$ above, order by order the effective action is given by

$$\begin{aligned}\Gamma_0[\Phi] &= S_{cl}[\phi] \\ \Gamma_1[\Phi] &= \frac{i\hbar}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\Phi=\phi_{cl}}\end{aligned} \tag{39}$$

Written out in detail,

$$\begin{aligned}\Gamma[\Phi] = &\int d^4x\Big[-\frac{1}{2}\partial_\mu\Phi_a\partial^\mu\Phi^a - \frac{m^2}{2}\Phi_a^2 - \frac{g}{4!}(\Phi_a^2)^2\Big] \\ &+ \frac{i\hbar}{2}\int d^4x d^4y \log\Big[(\Box - m^2)\delta_{ab} - \frac{g}{6}(\delta_{ab}\Phi_c^2 + 2\Phi_a\Phi_b)\Big](x,y)\delta^4(x,y).\end{aligned} \tag{40}$$

For a 1-d target space:

$$\begin{aligned}\Gamma[\Phi] = &\int d^4x\Big[-\frac{1}{2}(\partial\Phi)^2 - \frac{m^2}{2}\Phi^2 - \frac{g}{4!}(\Phi)^4\Big] \\ &+ \frac{i\hbar}{2}\,\mathrm{Tr}\log\Big[\Box - m^2 - \frac{g}{2}\Phi^2\Big]\end{aligned} \tag{41}$$

Where we recall that $\Phi \equiv 0$ is an extremum for $m^2 > 0$. We rewrite the first quantum correction as

$$\frac{i\hbar}{2}\ \mathrm{Tr}\ \log[\Box - m^2 - \frac{g}{2}\Phi^2] = \frac{i\hbar}{2}\ \mathrm{Tr}\ \log[\Box - m^2][1 - \frac{g}{2[\Box - m^2]}\Phi^2] \tag{42}$$

where by exploiting the properties of the logarithm, we can write this as

$$= \frac{i\hbar}{2}\ \mathrm{Tr}\ \log[\Box - m^2] + \frac{i\hbar}{2}\ \mathrm{Tr}\ \log[1 + \frac{i}{\Box - m^2}\frac{ig}{2}\Phi^2] \tag{43}$$

where the first term contains the usual Coleman-Weinberg correction to the effective potential, and where the second term generates derivative corrections to these. This can be seen by formally expanding the second term as

$$-\frac{i\hbar}{2}\sum_{n=1}^{\infty}\left(\frac{-ig}{2}\right)^n\frac{1}{n}\int d^4x d^4y\left[\left(\frac{i}{\Box - m^2}\right)\Phi^2\right]^n(x,y)\delta^4(x,y) \tag{44}$$

we find the more familiar derivative expansion (discussed in the slides) results by formally expanding the (free field) Green's function as

$$\frac{1}{\Box - m^2}(x,y) = \frac{-1}{m^2}\frac{1}{1 - \frac{\Box}{m^2}}(x,y) = -\delta^4(x,y)\frac{1}{m^2}\sum_{p=0}^{\infty}\left(\frac{\Box_x}{m^2}\right)^p \tag{45}$$

where now the scale m^2 plays the role that Λ did in our precious discussion.

Exercise: Take the expression (45) and explicitly expand (44) up to order $n = 2$. Eliminate redundant operators. What does the the resulting EFT look like up quartic order in derivatives?

The perhaps more familiar diagrammatic expansion (in position space) results by formally denoting $G_0 = \frac{i}{\Box - m^2}$, wherein the $n = 1$ term becomes

$$-\frac{\hbar}{4} g \int d^4x d^4y \delta^4(x,y) G_0(x,y) \Phi^2(y) \tag{46}$$

and similarly, the $n = 2$ term becomes

$$+\frac{i\hbar}{16} g^2 \int d^4x d^4y d^4z \; \delta^4(x,z) G_0(x,y) \Phi^2(y) G_0(y,z) \Phi^2(z) \tag{47}$$

Diagrammatically, these correspond to the following position space Feynman diagrams:

where each vertex is assigned a factor of g, and where the net effect of including higher and higher order corrections would be to keep adding two external legs to the loop integration. The fact that this action is valid only to one loop is due to the fact that there is only one independent internal variable to integrate over when one takes the trace in (44). *However, it remains to explicitly compute $G_0(x,y)$ and by extension, the second term in (43).* For this, we find it most convenient to introduce the method of the heat kernel, not least for its economy of method, but also because on a general curved background, it represents one of the most powerful methods to integrate out fields and to compute effective actions.

II. INTRODUCTION TO THE HEAT KERNEL I

For this part of the discussion, we switch to Euclidean signature and revert to units where $\hbar \equiv 1$). Recall that the object we are interested in calculating is (once we turn off external sources)

$$\mathrm{e}^{-W} = \det[-\Box + m^2(\psi)]^{-1/2} = \mathcal{N} \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{2}\int d^4x \, \phi[-\Box + m^2(\psi)]\phi} \tag{48}$$

where we allow for the mass to depend on some other field ψ, which could be an external background. Performing the same manipulations as before

$$\begin{aligned} W &= \frac{1}{2}\ln\,\det[-\Box + m^2(\psi)] \\ &= \frac{1}{2}\mathrm{Tr}\,\log[-\Box + m^2(\psi)] \\ &= \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}\;\mathrm{e}^{-s[-\Box + m^2(\psi)]} \end{aligned} \tag{49}$$

Where the last equation follows from the fact that

$$\begin{aligned} \lim_{\epsilon^2\to 0}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\mathrm{e}^{-sx} &= \lim_{\epsilon^2\to 0} -Ei[-x\epsilon^2] \\ &\approx (\infty) + \log[x] \end{aligned} \tag{50}$$

where the latter follows from asymptotic form of the exponential integral function, and where we drop a constant infinite term (more precisely, we absorb into an appropriate counterterm– more on this later). The factor ϵ^2 (of dimension length squared) is required to regulate the lower limit of the integral. Denoting the argument of the integrand in the final line of (49) as

$$G(x,x';s) := \theta(s)\langle x|\mathrm{e}^{-s[-\Box + m^2(\psi)]}|x'\rangle \tag{51}$$

so that

$$\mathrm{Tr}\;\mathrm{e}^{-s[-\Box + m^2(\psi)]} = \int d^4x\, G(x,x';s) \tag{52}$$

where the factor of $\theta(s)$ in (51) is to ensure that the trace is always convergent. We see from its definition that $G(x,x';s)$ satisfies the equation and normalization condition

$$\begin{aligned} [\partial_s - \Box_x + m^2(\psi)]G(x,x';s) &= \delta(s)\delta^4(x,x') \\ G(x,x';0) &= \delta^4(x,x'), \end{aligned} \tag{53}$$

which is the defining equation for the Green's function for the diffusion equation in 5-d, where we identify s with a fictitious time coordinate (recalling that the other four dimensions are Euclidean), justifying the terminology of 'Heat Kernel'. It's easy to see that its solution in flat space (in the limit where the ψ dependence of m^2 can be neglected) is given by

$$G(x,x';s) := \theta(s)\frac{\mathrm{e}^{-m^2(\psi)s}}{16\pi^2 s^2}\mathrm{e}^{-\frac{(x-x')^2}{4s}} \tag{54}$$

From (49) and (52), we see that the effective action is thus given by

$$W = \frac{1}{2}\int_{\epsilon:=\Lambda^{-2}}^{\infty}\frac{ds}{s}\int d^4x\, G(x,x;s) \tag{55}$$

Substituting the solution (54) into the above, we find that

$$W = \frac{1}{32\pi^2}\int d^4x\int_{\Lambda^{-2}}^{\infty}\frac{ds}{s^3}\mathrm{e}^{-m^2(\psi)s} \tag{56}$$

Evaluating the above integral, we find

$$W = \frac{1}{64\pi^2}\left[\Lambda^4 - m^4(\psi)\log\frac{m^2(\psi)}{\Lambda^2}\right] + ... \tag{57}$$

which is none other than the Coleman-Weinberg correction to the effective potential.

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[1] K. Huang, "Quantum field theory: From operators to path integrals," New York, USA: Wiley (1998) 426 p

[2] H. Kleinert, "Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets," World Scientific, Singapore, 2004

[3] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal field theory," New York, USA: Springer (1997) 890 p; Heat Kernel covered in Chapter 5.

[4] D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388**, 279 (2003) [hep-th/0306138].